Abstract

In the present work, we analyze the rotating shallow water equations including bottom topography using a space-time discontinuous Galerkin finite element method. The method results in non-linear equations per element, which are solved locally by establishing the element communication with a numerical HLLC flux. To deal with spurious oscillations around discontinuities, we employ a stabilization operator only around discontinuities using Krivodonova’s discontinuity detector. The numerical scheme is verified and validated by comparing numerical and exact solutions, and analyzing bore-vortex interactions. We conclude that the method is second order accurate in both space and time for linear polynomials, and correctly captures bore-vortex interactions.

Introduction

The rotating shallow water equations including topographic effects are a leading order model to study coastal hydrodynamics on several scales including intermediate scale rotational waves (100 km range) and breaking waves on beaches (1 km range). The shallow water model, in the absence of discontinuities, conserves potential vorticity, enstrophy and energy; and captures many interesting natural wave phenomena like flooding and drying at beaches, hurricanes approaching coastal zones and tsunamis.

In this paper, we analyze the rotating shallow water model using a space-time discontinuous Galerkin finite element method. This numerical method is originally developed in [1] to model inviscid compressible flows in a time dependent flow domain. In this method, we discretize and solve the shallow water model per space-time finite element locally by establishing the element communication through a numerical HLLC flux in the spatial direction and a numerical upstream flux in the time direction. The numerical discretization results in a set of coupled nonlinear equations which can be solved efficiently and locally, by adding a pseudo-time derivative and integrating them using a Runge-Kutta scheme until the solution reaches steady state in pseudo-time. We will use a multi-grid technique [1] for pseudo-time integration scheme, to improve the convergence acceleration.

The nonlinear shallow water equations can develop discontinuities in finite time. To limit spurious oscillations around discontinuities, we employ a stabilization operator (see [1]) only around discontinuities with the help of a discontinuity detector as in [2]. New in the present paper are the application of space-time method to rotating shallow water flows and the combination of a stabilization operator with the discontinuity detector. Furthermore, we present the numerical results for linear polynomials, show that the method is second order accurate in space-time and can accurately capture complex bore-vortex interactions.

Rotating Shallow Water Equations

Rotating shallow water equations including topographic terms can be concisely given in the index notation as

\[ \nabla \cdot F_i(U) = S_i \text{ in } \Omega, \]

where \( \nabla = (\partial_t, \partial_x, \partial_y) \) is the differential operator, \( U = (h, hu, hv) \) the state vector, \( h(x) \) the water depth, \( (u(x), v(x)) \) the velocity field,

\[ F_i(U) = \begin{pmatrix} h & hu & hv \\ hu & hu^2 + gh^2/2 & hv \\ hv & hw & hv^2 + gh^2/2 \end{pmatrix} \]

the flux vector, \( S_i = (0, fhv - gh\partial_x h_b, -fhu - gh\partial_y h_b) \) the source vector, \( f \) the Coriolis parameter, \( g \) the gravitational acceleration, \( h_b(x, y) \) the bottom topography and \( x = (t, x, y) \). Finally, we complete the system (1) with inflow, outflow or solid wall boundary conditions and initial conditions \( U(0, x, y) \).

Space-time Discontinuous Galerkin Method

In this section, we summarize the space-time discontinuous Galerkin method. At first, we tessellate the space-time domain with space-time elements \( K^i_n \) which are obtained by connecting the spatial elements in the time interval \( I_n = [t_n, t_{n+1}] \). Next, we define the finite element function space as:

\[ V_h^d := \left\{ V_h \in L^1(K^n) \left| V_h|_K \in (P^1(K))^d \right. \right\} \]

with \( P^1 \) the space of linear polynomials, \( d = \dim(V_h) \) and \( V_h \) the polynomial approximation defined as \( V_h := \)

\[ \]
\[ \sum_m \mathbf{V}_m \psi_m \] with \( \mathbf{V}_m \) the expansion coefficients and \( \psi_m \) the polynomial basis functions. The trace of the function \( \mathbf{V}_h \) on the element boundary \( \partial \mathcal{K}_h^n \) is defined as

\[
\mathbf{V}_h(x)|_{\partial \mathcal{K}_h^0} = \mathbf{V}^- := \lim_{\epsilon \to 0} \mathbf{V}_h(x - \epsilon \mathbf{n}_K)
\]

with \( \mathbf{n}_K \) as the outward unit normal vector of the boundary \( \partial \mathcal{K}_h^0 \). The discontinuous Galerkin weak formulation can now be obtained by multiplying (1) with the test function \( \mathbf{W}_h \in \mathcal{V}_h^d \) and integrating on each space-time element \( \mathcal{K}_h^k \): Find a \( \mathbf{U}_h \in \mathcal{V}_h^d \) such that for all \( \mathbf{W}_h \in \mathcal{V}_h^d \)

\[
\int_{\partial \mathcal{K}_h^0} \mathbf{n}_K \cdot \left( W^- \tilde{\mathbf{F}}_i (\mathbf{U}^-, \mathbf{U}^+, \mathbf{n}_K) \right) d(\partial \mathcal{K}) - \\
\int_{\mathcal{K}_h^k} \nabla W_{hi} \cdot \mathbf{F}_i (\mathbf{U}_h) \, d\mathcal{K} - \int_{\mathcal{K}_h^k} W_{hi} S_i \, d\mathcal{K} = 0 \quad (5)
\]

is satisfied. Note that, \( \tilde{\mathbf{F}}_i (\mathbf{U}^-, \mathbf{U}^+, \mathbf{n}_K) \) is the numerical flux, which directly depends on the discontinuous traces of \( \mathbf{U}_h \). To prevent numerical oscillations around discontinuities, we add a stabilization operator (see [1]) to the weak formulation (5) and use a discontinuity detector \( I_K \) for each space-time element (see [2]). The discontinuity detector acts as a switch indicating that an element is in a discontinuous region when \( I_K > 1 \) and in a smooth region otherwise. Thus, we apply the stabilization operator only to the elements in the discontinuous region.

**Numerical Results**

**Burger’s equation**

The one dimensional shallow water equations takes the form of Burger’s equation when one of the Riemann invariants is taken to be a constant. An implicit exact solution is constructed and compared with the numerical results. Our test indicates that the method is second order accurate before breaking occurs.

**Linear shallow water waves**

Linear exact shallow water solutions are considered in a rectangular domain with periodicity in \( x \) and solid walls along \( y \). These linear harmonic solutions are used as the initial conditions in the nonlinear numerical simulation (see Figure 1). Due to nonlinearity, harmonic waves start to break at some time which we can clearly see in Figure 2. We can also observe that there are no spurious oscillations around sharp gradients due to the application of stabilization operator.

**Flow over a conical hump**

Nonuniformity in the strength of the bores is an important source for the generation of potential vorticity

\[
(\text{PV}) = (\partial_x v - \partial_y u)/h,
\]

which was studied in [3] while in smooth regions \( \Pi \) is materially conserved, i.e.,

\[
\partial_t \Pi + u \partial_x \Pi + v \partial_y \Pi = 0.
\]

We consider a dam break and zero velocities as the initial conditions in the domain with a conical hump as bottom topography (see Figure 3). All the boundaries are considered to be solid except an open boundary at the exit. The initial dam at \( t = 0 \) collapses and a bore propagates downstream over the conical hump (see Figures 4 and 6). The interaction between the bore and the bottom topography generates PV which can be seen as two patches with opposite sign in Figure 5. The area of these patches increase with time (see Figure 7), but once the PV is generated it is materially conserved.

**Conclusions**

Rotating shallow water equations including topographic terms are numerically dealt with a space-time discontinuous Galerkin method. To avoid numerical oscillations around discontinuities, we use a stabilization operator only at discontinuities detected by the Krivodonova’s
detector. Our numerical results are second order accurate and show limited spurious oscillations. The numerical scheme is capable of generating potential vorticity due to bore-vortex interactions in accordance with the theoretical studies.

References